LECTURE 5: THE BLOCH-KATO TAMAGAWA NUMBER CONJECTURE

So far we have formulated the Beilinson conjectures, and seen that they hold in the case of number fields. We haven't discussed any higher dimensional examples. My intention for this final lecture had been to look at Bloch's calculation for $K_2(E)$ where E is a CM elliptic curve over $\mathbb{Q}$. However, a member of the audience requested that I talk a bit about the Tamagawa number conjecture of Bloch-Kato. It is Christmas, so I will. But first, for completeness, let me at least write down an incomplete list of cases where the Beilinson conjectures are known. I shall use the phrase "weak conjecture" to mean the statement of BC1 (resp. BC2, resp. BC3) with $H^i_{\mathcal{M}/\mathbb{Z}}(X, \mathbb{Q}(m))$ (resp. $H^{2j-1}_{\mathcal{M}/\mathbb{Z}}(X, \mathbb{Q}(j)) \oplus N^{j-1}(X)_{\mathbb{Q}}$, resp. $\mathrm{CH}^j_{\mathbb{Z}}(X)_0 \otimes \mathbb{Q}$) replaced by a suitable $\mathbb{Q}$-subspace.

(1) Number fields in full generality (Borel).
(2) The weak conjecture for $L(H^1(E), s)$ at $s = 2$ for CM elliptic curves over $\mathbb{Q}$ (Bloch, Beilinson). This was generalised to all $s \geq 2$ for CM elliptic curves over $\mathbb{Q}$, or for all elliptic curves over a number field F with CM by the ring of integers of an imaginary quadratic field K such that $K \supset F$, and such that $F(E_{\mathrm{tors}})$ is an abelian extension of K (Deninger).
(3) The weak conjecture for $L(H^1(C), s)$ at $s \geq 2$ for S a Shimura curve over $\mathbb{Q}$ (Ramakrishnan).
(4) The weak conjecture for $L(H^1(C), s)$ at $s \geq 2$ for C a compact modular curve over $\mathbb{Q}$ (Beilinson, Schappacher-Scholl).
(5) The weak conjecture for $L(H^2(C_1 \times C_2), s)$ at $s = 2$ for C_1, C_2 modular curves over $\mathbb{Q}$ (Beilinson).
(6) A weaker form than the weak conjecture for $L(H^2(X), s)$ at $s = 2$ for X the compactification of a Hilbert-Blumenthal surface over a real quadratic field (Ramakrishnan)
(7) The full conjecture for $L(H^1(E), s)$ at $s = 1$ (i.e. BSD) is known for elliptic curves E over $\mathbb{Q}$ with $\mathrm{ord}_{s=1} L(H^1(E), s) \in \{0, 1\}$ (Kolyvagin, Gross-Zagier).
(8) Some numerical verifications for specific curves and surfaces.

The list should be interpreted as telling you that very little is known beyond dimension 0 (remember how many conjectures we need before we can even start talking about the Beilinson conjectures!).

1. The Tamagawa number conjecture

This lecture will have to assume a bit more background than the previous ones. In particular, I assume that the students have some familiarity with a little p-adic Hodge theory. I won't talk about motives but be assured that there is a cohesive and compelling motivic picture underlying everything I am about to say. Also, one should talk about motives to treat e.g. modular forms.

1.1. **Notation.** Let K be a finite extension of $\mathbb{Q}_p$. Let ℓ be a prime and let V be a finite dimensional $\mathbb{Q}_\ell$-vector space with a continuous action of $G_K := \mathrm{Gal}(\overline{K}/K)$. Define the following subsets of the continuous Galois cohomology group $H^1(K,V) := H^1_{\mathrm{cont}}(G_K,V)$ as follows:

$$H^1_g(K,V) := \begin{cases} \ker(H^1(K,V) & \text{if } \ell \neq p \\ \ker(H^1(K,V) \to H^1(K, B_{\mathrm{dR}} \otimes V) & \text{if } \ell = p \end{cases}$$

and

$$H^1_f(K,V) := \begin{cases} \ker(H^1(K,V) \to H^1(K_{\mathrm{nr}},V) & \text{if } \ell \neq p \\ \ker(H^1(K,V) \to H^1(K, B_{\mathrm{cris}} \otimes V) & \text{if } \ell = p\,. \end{cases}$$

Then

$$H^1_f(K,V) \subseteq H^1_g(K,V) \subseteq H^1(K,V)\,.$$

For a free $\mathbb{Z}_\ell$-module of finite rank with a continuous G_K-action we define

$$H^1_*(K,T) := i^{-1}(H^1_*(K, T\otimes\mathbb{Q})), \quad * \in \{f,g\}$$

where $i : H^1(K,T) \to H^1(K,T\otimes\mathbb{Q})$ is the map induces by $T \hookrightarrow T\otimes\mathbb{Q}$.

For a free $\hat{\mathbb{Z}}$-module T of finite rank with a continuous G_K-action define

$$H^1_*(K,T) := \prod_\ell H^1_*(K, T\otimes_{\hat{\mathbb{Z}}} \mathbb{Z}_\ell), \quad * \in \{f,g\}\,.$$

Now let F be a number field. For a place v of F, write F_v for the completion of F at v. For a finite dimensional $\mathbb{Q}_\ell$-vector space with a continuous G_K-action, write

$$H^1_f(F,V) := \{x \in H^1(F,V) \,|\, x_v \in H^1_f(F_v,V) \text{ for all finite places } v \text{ of } F\}$$

and

$$H^1_g(F,V) := \{x \in H^1(F,V) \,|\, x_v \in H^1_g(F_v,V) \text{ for all finite places } v \text{ of } F \text{ and } x_v \in H^1_f(F_v,V) \text{ for all but finitely many } v\}$$

(where we view $\mathrm{Gal}(\overline{F}_v/F_v) \subset \mathrm{Gal}(\overline{F}/F)$). We define $H^1_*(F,T)$ in a similar way for T a free Λ-module of finite rank for $\Lambda \in \{\mathbb{Z}_\ell, \hat{\mathbb{Z}}, \mathbb{A}_f\}$ (where $\mathbb{A}_f = \hat{\mathbb{Z}}\otimes\mathbb{Q}$ is the ring of finite adèles over $\mathbb{Q}$).

1.2. **The conjecture.** Let X be a smooth projective variety over $\mathbb{Q}$ (for simplicity). Fix integers $m, r \geq 0$ such that $m \leq 2r-1$. Let

$$V := H^m_{\mathrm{sing}}(X(\mathbb{C}), \mathbb{Q}(r))\,.$$

Then $V \otimes \mathbb{A}_f \cong H^m_{\text{ét}}(X_{\overline{\mathbb{Q}}}, \mathbb{A}_f(r))$ is endowed with a continuous $G_{\mathbb{Q}}$-action. Let

$$D := H^m_{\mathrm{dR}}(X/\mathbb{Q})\,.$$

endowed with the filtration $D^\bullet$ given by $D^i := \mathrm{Fil}^{i+r} H^m_{\mathrm{dR}}(X/\mathbb{Q})$, $i \in \mathbb{Z}$ (i.e. the Hodge filtration shifted by r). There is an isomorphism of $\mathbb{R}$-vector spaces

$$D\otimes\mathbb{R} \xrightarrow{\sim} (V\otimes\mathbb{C})^+$$

and for each prime p there is an isomorphism of $\mathbb{Q}_p$-vector spaces

$$D\otimes\mathbb{Q}_p \xrightarrow{\sim} (B_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} V)^{G_{\mathbb{Q}_p}}$$

as Galois modules with a filtration (where the right hand side has the filtration coming from B_{dR}). The integer $w := m - 2r$ is called the "weight".

Set $\mathbb{Q}_\infty := \mathbb{R}$. For $p \in \{2,3,5,7,11,\ldots,\infty\}$ I shall sometimes write V_p, D_p etc for $V\otimes\mathbb{Q}_p$, $D\otimes\mathbb{Q}_p$ etc.

Let

$$\Phi := \begin{cases} H^{m+1}_{\mathcal{M}}(X, \mathbb{Q}(r)) & \text{if } m \neq 2r-1 \text{ (i.e. if } w \neq -1) \\ \mathrm{CH}^r(X)_0 \otimes \mathbb{Q} & \text{if } m = 2r-1 \text{ (i.e. if } w = -1)\,. \end{cases}$$

Let $\Phi_{/\mathbb{Z}} \subseteq \Phi$ denote the subspace of integral elements.

One should think of the Beilinson conjectures as being about the archimedean prime ∞, and the Bloch-Kato conjectures as incorporating the finite prime numbers p too. Indeed, the first part of the Beilinson conjectures says that the Beilinson regulator $r_{\mathcal{B}}$ induces an isomorphism

$$r_{\mathcal{B}} \otimes 1 : \Phi_{/\mathbb{Z}} \otimes \mathbb{R} \xrightarrow{\sim} H^{m+1}_{\mathcal{D}}(X_{\mathbb{R}}, \mathbb{R}(r))\,. \tag{1.2.1}$$

for $w \leq -3$ (and something similar when $w = -2, -1$). Let p be a prime number. Then a similar argument (due to Soulé, I think) as in Lecture 2 can be used to construct a map

$$H^{m+1}_{\mathcal{M}}(X, \mathbb{Q}(r)) \to H^{m+1}_{\text{ét}}(X, \mathbb{Q}_p(r))\,.$$

The Leray spectral sequence

$$E_2^{s,t} = H^s(\mathbb{Q}, H^t_{\text{ét}}(X_{\overline{\mathbb{Q}}}, \mathbb{Q}_p(r))) \Rightarrow H^{s+t}_{\text{ét}}(X, \mathbb{Q}_p(r))$$

degenerates at E_2, and the Weil conjectures (proved by Deligne) imply that $E_2^{0,t} = 0$ for $t \neq 2r$. For $r \neq \frac{m+1}{2}$ we get a map

$$H^{m+1}_{\text{ét}}(X, \mathbb{Q}_p(r)) \to H^1(\mathbb{Q}, H^m_{\text{ét}}(X_{\overline{\mathbb{Q}}}, \mathbb{Q}_p(r)))\,.$$

The p-adic regulator is the composition

$$r_p : H^{m+1}_{\mathcal{M}}(X, \mathbb{Q}(r)) \to H^{m+1}_{\text{ét}}(X, \mathbb{Q}_p(r)) \to H^1(\mathbb{Q}, H^m_{\text{ét}}(X_{\overline{\mathbb{Q}}}, \mathbb{Q}_p(r)))\,.$$

By work of Nekovář-Nizioł, the image of r_p is contained in $H^1_g(\mathbb{Q}, H^m_{\text{ét}}(X_{\overline{\mathbb{Q}}}, \mathbb{Q}_p(r)))$. The first part of the Bloch-Kato conjecture (independently conjectured by Jannsen) is the analogue (1.2.1) for finite primes:

Conjecture 1.3. (1) The p-adic regulators induce an isomorphism

$$\prod_p r_p \otimes 1 : \Phi \otimes \mathbb{A}_f \xrightarrow{\sim} H^1_g(\mathbb{Q}, V \otimes \mathbb{A}_f)\,.$$

(2) For each open set $U \subseteq \operatorname{Spec} \mathbb{Z}$ there exists a $\mathbb{Q}$-subspace $\Phi_U \subset \Psi$ such that

$$\Phi_U \otimes \mathbb{A}_f \xrightarrow{\sim} \{x \in H^1_g(\mathbb{Q}, V \otimes \mathbb{A}_f) \mid x_v \in H^1_f(\mathbb{Q}, V \otimes \mathbb{A}_f) \text{ if } v \in U\}$$

under the isomorphism in (1).

(3) If $m \neq 2r-1$ and X has a proper regular model $\mathcal{X}$ over U, then

$$\Phi_U = \mathrm{im}(H^{m+1}_{\mathcal{M}}(\mathcal{X}, \mathbb{Q}(r)) \to \Psi)\,.$$

If $m = 2r-1$ then $\Phi_U = \Psi$.

Remark 1.4. Note that in the case $U = \operatorname{Spec} \mathbb{Z}$, we should have $\Phi_{\operatorname{Spec} \mathbb{Z}} = \Phi_{/\mathbb{Z}}$. Then the conjecture asserts that

$$\prod_p r_p \otimes 1 : \Phi_{/\mathbb{Z}} \otimes \mathbb{A}_f \xrightarrow{\sim} H^1_f(\mathbb{Q}, V \otimes \mathbb{A}_f)\,.$$

From this and other good reasons, one should think of $H^1_f(\mathbb{Q}, V \otimes \mathbb{Q}_p)$ as being the analogue of $H^{m+1}_{\mathcal{D}}(X_{\mathbb{R}}, \mathbb{R}(r))$ for a finite prime $p < \infty$.

Example 1.5. Consider the case $r = m = 1$. Then

$$\Phi = \mathrm{CH}^1(X)_0 \otimes \mathbb{Q} = B(\mathbb{Q}) \otimes \mathbb{Q}$$

where $B = \mathrm{Pic}^0_{X/\mathbb{Q}}$ is the Picard variety of X. For $p < \infty$, the p-adic regulator r_p in the $m = r = 1$ case is just the map coming from the Kummer sequence. Then

$$B(\mathbb{Q}) \otimes \mathbb{A}_f \hookrightarrow H^1_g(\mathbb{Q}, V \otimes \mathbb{A}_f) \cong H^1_g(\mathbb{Q}, H^1_{\text{ét}}(B_{\overline{\mathbb{Q}}}, \mathbb{A}_f(1)))$$

is an isomorphism if and only if $|\text{Ш}(B)\{p\}| < \infty$ for all $p < \infty$.

Remark 1.6. It is also very interesting to study the p-adic regulator with torsion coefficients. For example, understanding the image of $H^3_{\mathcal{M}/\mathbb{Z}}(X, \mathbb{Z}(2)) \otimes \mathbb{Q}_p/\mathbb{Z}_p$ under r_p plays an important role in trying to establish finiteness of $\mathrm{CH}^2(X)\{p\}$.

Now that the the p-adic regulators and the Beilinson regulator have been put on an equal footing, we want to go beyond and look at the the residue of the L-function. Oversimplifying the idea of Bloch-Kato, one might say that they take inspiration from (among other places) the Birch and Swinnerton-Dyer conjecture, where a precise formula for the residue is predicted. Oversimplifying even further, one might say that the immediate problem you face is that the terms in the BSD formula use that abelian varieties are groups (e.g. the term $A(\mathbb{Q})_{\text{tors}}$ shows up). At least part of the idea is to define groups for X which are analogous to things like $A(\mathbb{Q}_p)$ and $A(\mathbb{Q})$. That is what we shall do now.

Let $M := H^m_{\text{sing}}(X(\mathbb{C}), \mathbb{Z}(r))/\text{torsion}$. Then M is a $\mathbb{Z}$-lattice M in V such that $M \otimes \hat{\mathbb{Z}} \subset V \otimes \mathbb{A}_f$ is stable under the action of $G_{\mathbb{Q}}$. Define

$$A(\mathbb{Q}_p) := \begin{cases} H^1_f(\mathbb{Q}_p, M \otimes \hat{\mathbb{Z}}) & \text{if } p < \infty \\ ((D \otimes \mathbb{C})/(D^0 \otimes \mathbb{C} + M))^+ & \text{if } p = \infty \end{cases}$$

(where the inclusion $M \subset D \otimes \mathbb{C}$ is via $D \otimes \mathbb{C} \simeq V \otimes \mathbb{C}$). Then $A(\mathbb{Q}_p)$ is compact (wrt the natural topology) for $p < \infty$ and $A(\mathbb{R})$ is locally compact. Under our hypotheses, the Bloch-Kato exponential at p is an isomorphism

$$\exp : D_p/D_p^0 \xrightarrow{\sim} H^1_f(\mathbb{Q}, V_p)$$

for all $p < \infty$. It turns out that these define a local isomorphism of topological groups:

$$\exp : D_p/D_p^0 \dashrightarrow A(\mathbb{Q}_p)\,.$$

For $p = \infty$ we just take the obvious map $D_\infty/D_\infty^0 \to A(\mathbb{R})$.

Fix a basis ω of $\det(D/D^0) \simeq \mathbb{Q}$. Then ω induces a basis of $\det(D_p/D_p^0) \simeq \mathbb{Q}_p$ and a Haar measure on each D_p/D_p^0 for each $p \leq \infty$, and hence a Haar measure $\mu_{p,\omega}$ on $A(\mathbb{Q}_p$ for each $p \leq \infty$ via the exp maps. It turns out that for $S = \{\text{bad reduction primes}\} \cup \{\infty\}$, the "size" $\mu_{p,\omega}(A(\mathbb{Q}_p))$ of $A(\mathbb{Q}_p)$ for $p \notin S$ turns out to be $P_p(H^i(X), 1)$, and in particular the product

$$\prod_{p \notin S} \mu_{p,\omega}(A(\mathbb{Q}_p))$$

converges if $w \leq -3$ (the region of absolute convergence of the L-function), so the product measure $\mu := \prod_{p \leq \infty} \mu_{p,\omega}$ on $\prod_{p \leq \infty} A(\mathbb{Q}_p)$ is well-defined. By the product formula, μ does not depend on the choice of ω.

Remark 1.7. Notice that the condition $w \leq -3$ corresponds to the BC1 setting, i.e. the region of absolute convergence for the L-function. We saw that this is the "easiest" setting. For weights $w = -2, -1$ one can define a measure μ on $\prod_{p \leq \infty} A(\mathbb{Q}_p)$ but it is a little more involved. From here on I will only look at the $w \leq -3$ case for simplicity.

Suppose that there exists a finite dimensional $\mathbb{Q}$-vector space Ψ with an isomorphism of $\mathbb{R}$-vector spaces

$$r_\infty : \Psi \otimes \mathbb{R} \xrightarrow{\sim} D_\infty/(D^0_\infty + V^+_\infty)$$

and an isomorphism of $\mathbb{A}_f$-modules

$$r_{\mathrm{Gal}} : \Psi \otimes \mathbb{A}_f \xrightarrow{\sim} H^1_f(\mathbb{Q}, V \otimes \mathbb{A}_f)\,.$$

Remark 1.8. BC1 says that we should be able to take $\Psi = \Phi_{/\mathbb{Z}}$ and $r_\infty = r_{\mathcal{B}}$. Then the isomorphism r_{Gal} is supposed to be the conjectured isomorphism $\prod_{p<\infty} r_p \otimes 1$ in Conjecture 1.3.

Let $A(\mathbb{Q})$ be the inverse image of $r_{\mathrm{Gal}}(\Psi)$ in $H^1_f(\mathbb{Q}, M \otimes \hat{\mathbb{Z}})$. Then $A(\mathbb{Q})$ is a finitely generated abelian group and there are the obvious maps $A(\mathbb{Q}) \to A(\mathbb{Q}_p)$ for $p \leq \infty$. Then the Tamagawa number is defined to be

$$\mathrm{Tam}(M) := \mu\left(\frac{\prod_{p\leq\infty} A(\mathbb{Q}_p)}{A(\mathbb{Q})}\right).$$

Also define

$$\text{Ш}(M) := \ker\left(\frac{H^1(\mathbb{Q}, M \otimes \mathbb{Q}/\mathbb{Z})}{A(\mathbb{Q}) \otimes \mathbb{Q}/\mathbb{Z}} \to \bigoplus_{p\leq\infty} \frac{H^1(\mathbb{Q}_p, M \otimes \mathbb{Q}/\mathbb{Z})}{A(\mathbb{Q}_p) \otimes \mathbb{Q}/\mathbb{Z}}\right).$$

Then the Bloch-Kato Tamagawa number conjecture asserts the following:

Conjecture 1.9. With the above notation (and still supposing $w \leq -3$):

(1) $|\text{Ш}(M)| < \infty$.

(2)
$$\mathrm{Tam}(M) = \frac{|H^0(\mathbb{Q}, M^* \otimes \mathbb{Q}/\mathbb{Z}(1))|}{|\text{Ш}(M)|}$$
$M^* := \mathrm{Hom}(M, \mathbb{Z})$.

Using that for $p \notin S = \{\text{bad reduction primes}\} \cup \{\infty\}$, we have $\mu_{p,\omega}(A(\mathbb{Q}_p)) = P_p(H^i(X), 1)$, we may rewrite the second part of the conjecture in the equivalent form that

$$L_S(V,0) = \frac{|\text{Ш}(M)|}{|H^0(\mathbb{Q}, M^* \otimes \mathbb{Q}/\mathbb{Z}(1))|} \cdot \mu_{\infty,\omega}(A(\mathbb{R})/A(\mathbb{Q})) \cdot \prod_{p \in S\setminus\{\infty\}} \mu_{p,\omega}(A(\mathbb{Q}_p))\,.$$

where

$$L_S(V,s) := \prod_{p\notin S} P_p(V, p^{-s})^{-1}$$

is the L-function of $H^m(X)(r)$ where we ignore the Euler factors at primes of bad reduction.

In particular, the Bloch-Kato Tamagawa number conjecture pins down the non-zero rational multiple in Beilinson's conjecture (but assumes a Beilinson conjecture statement as input). There is a conjecture for the near-central ($w = -2$) and central ($w = -1$) point too, but just like BC2 and BC3 they are a bit trickier to formulate. But, for example, the conjecture for $X = A$ an abelian variety and $m = r = 1$ is the same as the Birch and Swinnerton-Dyer conjecture.

1.10. **Example: The Reimann zeta function.** Let us look at the first possible example: Consider the case $X = \operatorname{Spec}\mathbb{Q}$, $m = 0$, $r \geq 2$ even (the calculation below breaks down into two cases: r even or r odd. For space reasons I choose to present only one case, and I choose the even one because it is easier!). Take $\omega = 1 \in \mathbb{Q} = H^0_{\mathrm{dR}}(\operatorname{Spec}\mathbb{Q})$ for the basis. For $p < \infty$ we have $A(\mathbb{Q}_p) := H^1_f(\mathbb{Q}_p, \hat{\mathbb{Z}}(r)) = H^1(\mathbb{Q}_p, \hat{\mathbb{Z}}(r))$ (by a p-adic Hodge theory calculation that I omit). Then Bloch-Kato compute that

$$\mu_{p,1}(A(\mathbb{Q}_p)) = |(r-1)|_p(1 - p^{-r})|H^0(\mathbb{Q}, \mathbb{Q}_p/\mathbb{Z}_p(1-r))|$$

and hence

$$\mu_p\left(\prod_{p<\infty} A(\mathbb{Q}_p)\right) := \prod_{p<\infty}(\mu_{p,1}(A(\mathbb{Q}_p))) = \frac{|H^0(\mathbb{Q}, \mathbb{Q}/\mathbb{Z}(1-r))|}{(r-1)!\zeta(r)}.$$

If r is even then $A(\mathbb{R}) = \mathbb{R}/(2\pi)^r\mathbb{Z}$ so

$$\begin{aligned}
\operatorname{Tam}(\mathbb{Z}(r)) &= \frac{\mu_p\left(\prod_{p\leq\infty} A(\mathbb{Q}_p)\right)}{|A(\mathbb{Q})|} \\
&= \frac{|H^0(\mathbb{Q}, \mathbb{Q}/\mathbb{Z}(1-r))| \cdot (2\pi)^r}{|H^0(\mathbb{Q}, \mathbb{Q}/\mathbb{Z}(r))| \cdot (r-1)! \cdot \zeta(r)} \\
&= \pm\frac{|H^0(\mathbb{Q}, \mathbb{Q}/\mathbb{Z}(1-r))|}{|H^0(\mathbb{Q}, \mathbb{Q}/\mathbb{Z}(r))|} \cdot \frac{2}{\zeta(1-r)}.
\end{aligned}$$

One can check (using Tate duality) that for $p \neq 2$ (and r even) we have

$$Ш(\mathbb{Z}(r))\{p\} \cong H^1(\mathbb{Z}[1/p], \mathbb{Q}_p/\mathbb{Z}_p(r)).$$

Therefore, up to a power of 2, the Tamagawa number conjecture for $X = \operatorname{Spec}\mathbb{Q}$, $m = 0$, $r \geq 2$ even is equivalent to the statement that

$$\zeta(1-r) = \pm\prod_{p<\infty}\frac{|H^1(\mathbb{Z}[1/p], \mathbb{Q}_p/\mathbb{Z}_p(r))|}{|H^0(\mathbb{Z}[1/p], \mathbb{Q}_p/\mathbb{Z}_p(r))|}$$

for r even. But this equality was proved by Mazur-Wiles as a consequence of their proof of the Iwasawa main conjecture for $\mathbb{Q}$.